Research Statement

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The height-to-radical transform

The goal is to define a height-to-radical transform which gives the radical as a degenerate height. The first step is to define the radical in a conceptual way with a contribution from the archimedean valuations, in such a way that the ABC conjecture reads $h \leq r$, without an error term. This is worked out for meromorphic functions in *The ABC Conjecture for Meromorphic Functions* (to be submitted). I applied for an NSF grant to fund this project.

To learn the formalism, I wrote the paper *The ABC conjecture implies Vojta's Height Inequality for Curves* (J. Number Theory **95** (2002), 289–302). In it I prove this using Belyĭ's construction, following Elkies and Bombieri. I learned the Arakelov definition of the height and how it relates to the radical. This paper has a less technical predecessor: *The ABC Conjecture implies Roth's Theorem and Mordell's Conjecture* (Matemática Contemporânea **16** (1999), 45–72).

I also wrote *A Lower Bound in the ABC Conjecture* (J. Number Theory **82** (2000), 91–95) to understand from a probabilistic point of view the strongest possible form of the ABC conjecture.

The Riemann hypothesis for curves over a finite field

This is related to the previous project: Connes' approach to the Riemann hypothesis provides a one-dimensional noncommutative space, and the two-dimensional analogue of this space has applications to the ABC conjecture.

The first step (the second step of the previous project) is to understand Haran's theory (which is related to Connes' approach to the Riemann hypothesis) in his paper *On Riemann's zeta-function* (*Dynamical, Spectral and Arithmetic Zeta Functions,* M.L. Lapidus and M.v.F., eds., Contemporary Mathematics **290**, AMS, Providence, RI, 2001). I am writing a research monograph about the function field case to accomplish this, *The Riemann Hypothesis for Function Fields (Frobenius flow and shift operators),* and in February 2007 I was invited by the IHES (Bures-sur-Yvette, France) to work on this with Alain Connes.

Two papers have been derived from this monograph, *The Zeta Function of a Function Field,* and *The Riemann Hypothesis for Function Fields,* which have both been submitted recently.

Complex dimensions of fractals

The goal is to develop a theory of complex dimensions of fractal objects.

We have made an extensive study of the complex dimensions of self-similar nonlattice strings, which are quasi-periodic (*Complex Dimensions of Self-Similar*

Fractal Strings and Diophantine Approximation, M.L. Lapidus and M.v.F., Experimental Mathematics **12** (2003), 41–69). I learned the computational aspects from this.

The next goal is to view the complex dimensions of a nonlattice string as the one-dimensional section of a *periodic* higher-dimensional pattern. This is the subject of ongoing research.

The one-dimensional theory is explained in the book *Fractal Geometry, Complex Dimensions and Zeta Functions (geometry and spectra of fractal strings),* M.L. Lapidus and M.v.F., Springer Monographs in Mathematics, Springer–Verlag, 2006.

Hecke algebras in number theory

We study Connes' approach to the Riemann hypothesis for general number fields. Our first goal is to find the right Hecke algebra. A long-term goal is to use this construction to formulate class field theory, and to give an interpretation of the full idele class group.

Let K be a number field with ring of integers $\mathcal{O}$. Denote the units of $\mathcal{O}$ by $\mathcal{O}^*$, and the multiplicative semigroup by $\mathcal{O}^\times$. In the paper *Phase transitions on Hecke C*-algebras and class-field theory* (M. Laca and M.v.F., J. reine angew. Math. **595** (2006), 25–53), we study the Hecke C*-algebra of functions

$$f(\gamma), \qquad \gamma = \begin{pmatrix} 1 & y \\ 0 & x \end{pmatrix}$$

for $y \in K$ and $x \in K^*$, invariant under the action of the ring of integers

$$f(\alpha\gamma) = f(\gamma\alpha) = f(\gamma)$$

for every $\alpha = \begin{pmatrix} 1 & a \\ 0 & u \end{pmatrix}$, $a \in \mathcal{O}$, $u \in \mathcal{O}^*$. Multiplication is defined by convolution

$$f * g(\gamma) = \sum_{\delta} f(\gamma\delta^{-1})g(\delta)$$

where the summation is over the left cosets

$$\left\{ \begin{pmatrix} 1 & a \\ 0 & u \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & x \end{pmatrix} : a \in \mathcal{O}, u \in \mathcal{O}^* \right\}.$$

This algebra admits an action of the multiplicative semigroup $\mathcal{O}^\times/\mathcal{O}^*$. We find the KMS$_\beta$ states ($\beta \in [0,\infty]$). The partition function is the Dedekind zeta function of K.

The symmetry group of the KMS states is the Weil-group, which is canonically isomorphic to the Galois group of the maximal abelian extension of K. However, the action on KMS states corresponds to the action of the Galois group only when K is the field of rational numbers.

I am invited by M. Laca to Victoria in summer 2007 to continue our research.