

ARITHMETIC PROGRESSIONS OF ZEROS OF THE RIEMANN ZETA FUNCTION

MACHIEL VAN FRANKENHUIJSEN

ABSTRACT. If the Riemann zeta function vanishes at each point of the finite arithmetic progression $\{D + in\mathbf{p}\}_{0 < |n| < N}$ ($D \geq 1/2$, $\mathbf{p} > 0$), then $N < 13\mathbf{p}$ if $D = 1/2$, and $N < \mathbf{p}^{1/D-1+o(1)}$ in general.

1. INTRODUCTION

In 1954, Putnam showed that the Riemann zeta function does not have an infinite vertical arithmetic progression of zeros (or even of ‘approximate zeros’; see [P1, 2]). His proof depends on unique factorization of the integers and is hard to generalize to other zeta functions. In 1997, Lapidus and the author found a new proof of Putnam’s theorem, which extends to a large class of zeta functions and L -series, see [LvF, Ch. 9]. It remained a question how long an arithmetic progression of zeros can be. This question was answered in 1998 by Watkins, for shifted progressions of zeros $\{D + in\mathbf{p}\}_{0 < |n| < N}$ ($D \in \mathbb{C}$, $\operatorname{Re} D = d \in (0, 1)$) of any Dirichlet L -series. He proves in [W] (see also [vFW]) that if $L(D + in\mathbf{p}, \chi) = 0$ for all n , $0 < |n| < N$, then $\log N < (\pi^{-1} + o(1))(d + 1)^{-1} d^{-2} \mathbf{p} \log^2 \mathbf{p}$.

Remark 1.1. This type of results is often regarded ‘folklore’ among number theorists, but I know of no other references than those cited above. The work of Odlyzko and te Riele [OtR] on the Mertens conjecture contains a lot of interesting related information. Also Stark’s work [S] is related.

In the present paper, we return to the Riemann zeta function and arithmetic progressions starting on the real axis, i.e., D is real in the interval $(0, 1)$. However, it might be possible to extend the method of proof to shifted arithmetic progressions and to L -series. Our result is

Theorem 1.2. *Let $\mathbf{p} > 0$, $N \geq 2$, and suppose $\zeta(D + in\mathbf{p}) = 0$ for $0 < |n| < N$. Then*

$$N < 60 \left(\frac{\mathbf{p}}{2\pi} \right)^{\frac{1}{D}-1} \log \mathbf{p}.$$

Moreover, $N < 13\mathbf{p}$ if $D = 1/2$ and $N < 80(\mathbf{p}/2\pi)^{1/D-1}$ if $D < 0.96$.

Without loss of generality, we can assume that $D \geq 1/2$. Thus the length of an arithmetic progression is bounded by $O(\mathbf{p})$ for $D = 1/2$, and by $o(\mathbf{p})$ for $D > 1/2$.

Remark 1.3. One generally conjectures that this theorem remains valid with the conclusion that $D = 1/2$ and $N \leq 2$ (i.e., the Riemann hypothesis and there are no arithmetic progressions of zeros on the line $\operatorname{Re} s = 1/2$). Indeed, no such progression of zeros of the Riemann zeta function, even of length two (i.e., $1/2 + i\mathbf{p}$ and $1/2 + 2i\mathbf{p}$ are both zeros of $\zeta(s)$), and no zero off the line $\operatorname{Re} s = 1/2$, has ever

been found. This follows from the tables of zeros of the Riemann zeta function, which allow me to verify this numerically up to $\mathbf{p} = 37,460$ (the largest zero in my table is $1/2 + 74920.8i$; see [O]). Throughout this paper, we will assume, whenever necessary, that $\mathbf{p} > 37,000$. See also footnote 1.

Recently, I obtained a larger table from Odlyzko, which would allow one to assume $\mathbf{p} > 1.13 \cdot 10^6$, but this does not substantially improve the bounds in this paper.¹ For the 924,280th zero in this table, $1/2 + it = 1/2 + 558652.035125523i$, the point $1/2 + 2it$ is very close to the 1,971,817th zero, $1/2 + 1117304.070251415i$. However, there are still three significant digits to distinguish these points from each other. This is the closest approximation to an arithmetic progression (of length two) of zeros in my table.

See [LvF, Ch. 10] for some conjectures and examples regarding the relationship between the number of poles of a Dirichlet series and the maximal possible length of an arithmetic progression of zeros.

We close this introduction by giving an overview of the proof of Theorem 1.2. We define the function $T(x)$ by

$$(1.1) \quad T(x) = \frac{1}{\log a} \sum_{|n| < N} \left(1 - \frac{|n|}{N}\right) \frac{x^{D+in\mathbf{p}} - 1}{D + in\mathbf{p}},$$

for $x \geq 1$, and $T(x) = 0$ for $x \leq 1$ (we write $a = e^{2\pi/\mathbf{p}}$). This function is differentiable with positive derivative given by $T'(x) = \frac{1}{\log a} x^{D-1} K_N(\log_a x)$ (we write $\log_a x = \log x / \log a$ for the logarithm of x with base a ; also see Definition 2.1 and Proposition 2.2 for the Fejer kernel K_N). Therefore

$$(1.2) \quad T(x) = \frac{1}{\log a} \int_1^x t^D K_N(\log_a t) \frac{dt}{t} = \int_0^{\log_a x} a^{tD} K_N(t) dt$$

is increasing, and it is sharply increasing at $x = a^m$, an integral power of a , if N is large.

We also consider the following function associated with $T(x)$,

$$(1.3) \quad \nu_T(x) = \sum_{k \leq x} T(x/k).$$

Thus, $\nu_T(x)$ increases at least as sharply as the first term $T(x)$ at $x = a^m$. The main auxiliary result of this paper is an explicit formula for this function with an exact expression for the error term,

$$(1.4) \quad \nu_T(x) = c_T x + \frac{1}{\log a} \sum_{|n| < N} \left(1 - \frac{|n|}{N}\right) \frac{x^{D+in\mathbf{p}}}{D + in\mathbf{p}} \zeta(D + in\mathbf{p}) + O(1),$$

where $O(1)$ denotes the bounded function given in Lemma 3.1 below, and

$$(1.5) \quad c_T = \frac{1}{\log a} \sum_{|n| < N} \left(1 - \frac{|n|}{N}\right) \frac{1}{1 - D - in\mathbf{p}} = \int_0^\infty a^{(D-1)t} K_N(t) dt.$$

If we assume that the Riemann zeta function has an arithmetic progression of zeros of length $N - 1$, that is, $\zeta(D + in\mathbf{p}) = 0$ for all n , $0 < |n| < N$, then by the explicit

¹The bound $N < 13\mathbf{p}$ for zeros on the line $\operatorname{Re} s = 1/2$ holds for $\mathbf{p} > 44,000$. For all other bounds, $\mathbf{p} > 37,000$ suffices.

formula (1.4), the increase of $\nu_T(x)$ at $x = a^m$ is small. We thus obtain a bound for N .

Remark 1.4. In the language of [LvF, Ch. 8, 9], the function $T(x)$ is the geometric counting function of the so-called *truncated generalized Cantor string*. It is defined by the ‘explicit formula’ (1.1). The function $\nu_T(x)$ of (1.3) is the counting function of the spectrum of this fractal string. Formula (1.5) gives its volume, and $c_T x$ is called the ‘Weyl-term’ in Weyl’s asymptotic law (1.4).

In the next section, we give an expression for the Riemann zeta function that will be used to derive (1.4) above. We also establish a zero-free region for the Riemann zeta function, which allows us to rewrite our result in a more useful form.

2. A ZERO-FREE REGION FOR $\zeta(s)$

The following function approximates the sum of delta functions $\sum_{n \in \mathbb{Z}} \delta_{\{n\}}$ (the ‘function’ $\delta_{\{n\}}$ denotes the distribution $\int_{-\infty}^{\infty} \delta_{\{n\}}(x) f(x) dx = f(n)$):

Definition 2.1. Let $N > 0$. The *Fejer kernel* is the function

$$K_N(x) = \sum_{|n| < N} \left(1 - \frac{|n|}{N}\right) e^{2\pi i n x} = \sum_{|n| < N} c_n e^{2\pi i n x},$$

for $x \in \mathbb{R}$. We write $c_n = 1 - |n|/N$ for the coefficients of K_N , N being fixed.

Proposition 2.2. *The function K_N has most of its mass concentrated around the integers:*

- (i) $K_N(x) \geq 0$ for all $x \in \mathbb{R}$,
- (ii) the total mass $\int_0^1 K_N(x) dx = 1$,
- (iii) $\int_0^{1/N} K_N(x) dx > \int_0^1 \left(\frac{\sin \pi t}{\pi t}\right)^2 dt = C_1 > 9/20$.

Proof. It is well known that $K_N(x) = \frac{1}{N} \left(\frac{\sin \pi N x}{\sin \pi x}\right)^2$, which proves (i). Also (ii) is clear. For (iii), we use $\sin^2(\pi x) \leq (\pi x)^2$ and the substitution $Nx = t$. Using Maple, we find that the integral is slightly larger than 0.45. \square

Let $B_1(x) = x - \frac{1}{2}$ be the first Bernoulli polynomial, and let $\{x\} = x - [x]$ denote the fractional part of the real number x . We have $B_1'(\{x\}) = 1 - \sum_{n \in \mathbb{Z}} \delta_{\{n\}}$. Therefore, if the function f is continuously differentiable and f and f' are integrable on $[x, \infty)$, we obtain

$$\begin{aligned} \int_x^\infty f(t) dt - \sum_{k=[x]+1}^\infty f(k) &= \int_{x+}^\infty f(t) dB_1(\{t\}) \\ &= -f(x)B_1(\{x\}) - \int_x^\infty f'(t)B_1(\{t\}) dt, \end{aligned}$$

using that $\{x+\} = \{x\}$. We apply this to $f(t) = \frac{1}{s}(x/t)^s$ to derive the following lemma, which is a generalization of [T, Eq. (3.5.3)]:

Lemma 2.3. *For $\operatorname{Re} s > 0$ and $x > 0$,*

$$\sum_{k=1}^{[x]} \frac{(x/k)^s - 1}{s} = \frac{x}{1-s} + \frac{x^s}{s} \zeta(s) + \frac{1}{2s} + \int_1^\infty B_1(\{xt\}) t^{-s-1} dt.$$

Here, $\zeta(s) = \sum_{k=1}^{\infty} k^{-s}$ denotes the Riemann zeta function. It is well known that this function has a meromorphic continuation to the entire complex plane, and that the completed zeta function $\zeta_{\mathbb{Z}}(s) = \Gamma(s/2)\pi^{-s/2}\zeta(s)$ satisfies the functional equation $\zeta_{\mathbb{Z}}(s) = \zeta_{\mathbb{Z}}(1-s)$. The completed zeta function has simple poles at 0 and 1, it has no zeros on the real line, and all its zeros ρ lie in the strip $0 < \operatorname{Re} \rho < 1$, symmetric about the real axis and the line $\operatorname{Re} s = 1/2$. Pairing a zero with its complex conjugate, we can write the Weierstrass product for this function as

$$(2.1) \quad \zeta_{\mathbb{Z}}(s) = \frac{C_2}{s(1-s)} \prod_{\operatorname{Im} \rho > 0} \left(1 - \frac{s-1/2}{\rho-1/2}\right) \left(1 - \frac{s-1/2}{\bar{\rho}-1/2}\right).$$

Next, we derive a zero-free region for the Riemann zeta function, by a slight improvement of a classical argument. The result below is not best possible, see [I, footnote 3, p. 66]; we need it for the explicit constants.

Let $\frac{\Gamma'}{\Gamma}(s) = \log s - \frac{1}{2s} + \int_0^{\infty} B_1(\{t\})(t+s)^{-2} dt$ be the derivative of Stirling's formula for the Gamma function of Euler (see [T,I]). It follows that for $\operatorname{Re} z > 0$,

$$(2.2) \quad \frac{\Gamma'}{\Gamma}(z) + \frac{\Gamma'}{\Gamma}(\bar{z}) = \log |z|^2 - \frac{\operatorname{Re} z}{|z|^2} + b(z),$$

where $b(z)$ is real and $|b(z)| \leq 1/|\operatorname{Im} z|$. Since $\frac{1}{2} \frac{\Gamma'}{\Gamma}(\frac{s}{2}) + \frac{1}{s} = \frac{1}{2} \frac{\Gamma'}{\Gamma}(\frac{s}{2} + 1)$, and using (2.1), we obtain the logarithmic derivative

$$(2.3) \quad \frac{1}{2} \frac{\Gamma'}{\Gamma} \left(\frac{s}{2} + 1 \right) - \frac{1}{2} \log \pi + \frac{\zeta'}{\zeta}(s) = -\frac{1}{s-1} + \sum_{\operatorname{Im} \rho > 0} \frac{1}{s-\rho} + \frac{1}{s-\bar{\rho}},$$

convergent for $s \neq 1$ or any of the zeros ρ or $\bar{\rho}$.

Theorem 2.4. *Suppose $\mathbf{p} > 0$ and $\zeta(D + i\mathbf{p}) = 0$. Then $\frac{1}{1-D} < 18 \log \mathbf{p} - 27$.*

Proof. Let $K(x) = \sum_n a_n e^{2\pi i n x}$ be a kernel, satisfying (i) $K(x) \geq 0$, (ii) $a_n \geq 0$ (hence $a_{-n} = a_n$ and K is even), and (iii) $2a_1 > a_0$. Using the Dirichlet series² $-\frac{\zeta'}{\zeta}(s) = \sum_{k=1}^{\infty} \Lambda(k)k^{-s}$ and (i), we obtain for $\sigma > 1$ that

$$-\sum_n a_n \frac{\zeta'}{\zeta}(\sigma + in\mathbf{p}) = \sum_{k=1}^{\infty} \Lambda(k)k^{-\sigma} K(\log_a k) \geq 0,$$

with $a = e^{2\pi/\mathbf{p}}$. By (ii), since $a_{-n} = a_n$ and $\sigma > 1 > \operatorname{Re} \rho$, we have for a fixed nontrivial zero ρ of the Riemann zeta function that

$$(2.4) \quad \sum_n a_n \left(\frac{1}{\sigma + in\mathbf{p} - \rho} + \frac{1}{\sigma + in\mathbf{p} - \bar{\rho}} \right) = \sum_n \frac{2a_n(\sigma - \operatorname{Re} \rho)}{|\sigma + in\mathbf{p} - \rho|^2} > 0.$$

Thus the sum over the zeros $\rho \neq D + i\mathbf{p}$ on the right-hand side of formula (2.3) is positive. We also apply (2.4) to the zero $\rho = D + i\mathbf{p}$, but then we single out the term for $n = 1$ to obtain

$$\frac{2a_1}{\sigma - D} + \sum_{n \neq 1} \frac{2a_n(\sigma - D)}{|\sigma - D + i(n-1)\mathbf{p}|^2}.$$

²The von Mangoldt function is defined by $\Lambda(p^l) = \log p$ for a prime power, and $\Lambda(k) = 0$ if k is not a power of a prime number.

The sum is positive, and the first term is very large when σ is close to D . We thus obtain from (2.3)

$$\sum_n \frac{a_n \Gamma'}{2 \Gamma} \left(\frac{\sigma + in\mathbf{p}}{2} + 1 \right) - \frac{K(0)}{2} \log \pi > \frac{2a_1}{\sigma - D} + \frac{a_0}{1 - \sigma} - \sum_{n \geq 1} \frac{2(\sigma - 1)a_n}{(\sigma - 1)^2 + n^2 \mathbf{p}^2}.$$

By (iii), $2a_1(\sigma - D)^{-1} + a_0(1 - \sigma)^{-1}$ has a maximum $(\sqrt{2a_1} - \sqrt{a_0})^2 / (1 - D)$, at $\sigma = 1 + (1 - D)(\sqrt{2a_1/a_0} - 1)^{-1}$. We obtain, using (2.2) and $2 \sum_{n \geq 1} a_n = K(0) - a_0$,

$$\begin{aligned} \frac{2(\sqrt{2a_1} - \sqrt{a_0})^2}{1 - D} &< \sum_{n \geq 1} a_n \log((\sigma + 2)^2 + n^2 \mathbf{p}^2) + a_0 \frac{\Gamma'}{\Gamma} \left(\frac{\sigma}{2} + 1 \right) + \sum_{n \geq 1} \frac{2a_n}{n\mathbf{p}} \\ &- \sum_{n \geq 1} \frac{2(\sigma + 2)a_n}{(\sigma + 2)^2 + n^2 \mathbf{p}^2} - K(0) \log 2\pi + a_0 \log 2 + \sum_{n \geq 1} \frac{4(\sigma - 1)a_n}{(\sigma - 1)^2 + n^2 \mathbf{p}^2}. \end{aligned}$$

The first sum on the right-hand side is of order $2 \sum_{n \geq 1} a_n \log \mathbf{p}$ as $\mathbf{p} \rightarrow \infty$, hence the optimal choice of the coefficients a_n is when $(\sqrt{2a_1} - \sqrt{a_0})^{-2} \sum_{n \geq 1} a_n$ is minimal. Ingham takes the function $(1 + \cos(2\pi x))^2$, but a better choice is $(1 + \cos(2\pi x))^6$. Then $\sigma < 2.61652$. We evaluate each increasing term on the right at this value and each decreasing term at $\sigma = 1$. Since $\mathbf{p} > 37,000$ if $D > 1/2$ by Remark 1.3, we obtain $1/(1 - D) < 17.9412 \log \mathbf{p} - 27.710$. This establishes the zero-free region for the Riemann zeta function. \square

3. THE EXPLICIT FORMULA FOR ν_T

Recall the notation $c_n = 1 - |n|/N$. We derive an explicit formula (Weyl's asymptotic law) for the function $\nu_T(x)$ of formula (1.3).

Lemma 3.1. *We have the following explicit formula for $\nu_T(x)$:*

$$\begin{aligned} \nu_T(x) &= c_T x + \frac{1}{\log a} \sum_{|n| < N} c_n \frac{x^{D+in\mathbf{p}}}{D+in\mathbf{p}} \zeta(D+in\mathbf{p}) \\ &+ \frac{1}{2 \log a} \sum_{|n| < N} \frac{c_n}{D+in\mathbf{p}} + \int_0^\infty B_1(\{xa^t\}) a^{-tD} K_N(t) dt. \end{aligned}$$

Proof. We use formula (1.1) and Lemma 2.3 to compute $T(x/k)$ in (1.3). Thus we obtain

$$\begin{aligned} \nu_T(x) &= \frac{1}{\log a} \sum_{|n| < N} c_n \left(\frac{x}{1 - D - in\mathbf{p}} + \frac{x^{D+in\mathbf{p}}}{D+in\mathbf{p}} \zeta(D+in\mathbf{p}) \right. \\ &\quad \left. + \frac{1}{2(D+in\mathbf{p})} + \int_1^\infty B_1(\{xt\}) t^{-D-in\mathbf{p}-1} dt \right). \end{aligned}$$

By (1.5), the first term gives the Weyl-term $c_T x$. The lemma follows after making the substitution $t \rightarrow a^t$ in the integral. \square

4. ZEROS IN ARITHMETIC PROGRESSION

We will obtain a bound for N by first showing that the function $\nu_T(x)$ increases by a large amount as x increases from a^m to $a^{m+\varepsilon}$:

Lemma 4.1. *Let $m \in \{0, 1, 2, \dots\}$ and $\varepsilon = 1/N$. Then*

$$\nu_T(a^{m+\varepsilon}) - \nu_T(a^m) > C_1 a^{mD}.$$

Proof. Since $\nu_T(x) = \sum_{k=1}^{\infty} T(x/k)$, and $T(x)$ is increasing, we have for $y > x$, by (1.2), that

$$\nu_T(y) - \nu_T(x) \geq T(y) - T(x) = \int_{\log_a x}^{\log_a y} a^{tD} K_N(t) dt.$$

For $x = a^m$ and $y = a^{m+\varepsilon}$, we obtain an increase of $a^{mD} \int_0^\varepsilon K_N(t) dt > C_1 a^{mD}$, by Proposition 2.2(iii). \square

Lemma 4.2. *Let $C_3 = e^{2\zeta(2)}$. For $d \in (0, 1)$ we have*

$$\sum_{|n| < N} \frac{c_n}{d + in\mathbf{p}} < C_3^{1/\mathbf{p}^2} \frac{1}{d}.$$

Proof. $c_n \left(\frac{1}{d - in\mathbf{p}} + \frac{1}{d + in\mathbf{p}} \right) < 2n^{-2}/\mathbf{p}^2$ and $\sum_{n=1}^{\infty} n^{-2} = \zeta(2)$. \square

Let $T(x)$ be given by (1.1) and (1.2) above. Recall the coefficient c_T given by (1.5). We deduce for $d = 1 - D$ that

$$(4.1) \quad c_T \log a < C_3^{1/\mathbf{p}^2} \frac{1}{1 - D}.$$

Next we assume that $\zeta(D + in\mathbf{p}) = 0$ for $0 < |n| \leq N - 1$, and show that $\nu_T(x)$ does not increase by much.

Lemma 4.3. *Assume $\zeta(D + in\mathbf{p}) = 0$ for $0 < |n| \leq N - 1$. For $m \in \{0, 1, 2, \dots\}$ and $\varepsilon = 1/N$, we have*

$$\frac{\nu_T(a^{m+\varepsilon}) - \nu_T(a^m)}{\varepsilon} < C_3^{1/\mathbf{p}^2} \frac{a^{m+\varepsilon}}{1 - D} + a^{mD} \zeta(D) + C_3^{1/\mathbf{p}^2} \frac{N}{D \log a}.$$

Proof. By Lemma 3.1 we have

$$(4.2) \quad \nu_T(a^{m+\varepsilon}) - \nu_T(a^m) = a^m (a^\varepsilon - 1) c_T + \frac{a^{mD}}{\log a} \frac{a^\varepsilon D - 1}{D} \zeta(D) \\ + \int_0^\infty (B_1(\{a^{m+\varepsilon+t}\}) - B_1(\{a^{m+t}\})) a^{-tD} K_N(t) dt.$$

We use $|B_1(\{x\})| \leq \frac{1}{2}$ and $K_N(t) \geq 0$ to estimate the integral by $\int_0^\infty a^{-tD} K_N(t) dt$. This integral evaluates to

$$\frac{1}{\log a} \sum_{|n| < N} \frac{c_n}{D + in\mathbf{p}} < C_3^{1/\mathbf{p}^2} \frac{1}{D \log a},$$

by Lemma 4.2. We also estimate c_T by (4.1). Then we multiply by $N = 1/\varepsilon$ and estimate $(a^\varepsilon - 1)/\varepsilon < a^\varepsilon \log a$ and $(a^\varepsilon D - 1)/\varepsilon D > \log a$ by the Mean Value Theorem (recall that $\zeta(D) < 0$). \square

To complete the proof of Theorem 1.2, we need one final estimate:

Lemma 4.4. *For $\mathbf{p} > 37,000$, we have $1 - aD > (1 - D)e^{-120(\log \mathbf{p} - 3/2)/\mathbf{p}}$.*

Proof. Use $a = e^{2\pi/\mathbf{p}}$ to estimate $\frac{1-aD}{1-D} = 1 - (a-1)\frac{D}{1-D}$. Then use Theorem 2.4. \square

Proof of Theorem 1.2. Write $A = C_1 N(1 - D) - \zeta(D)(1 - D)$ and $B = a^\varepsilon C_3^{1/\mathbf{p}^2}$. Combining Lemma 4.1 and 4.3, we obtain, since $\zeta(D) < 0$ and $\log a = 2\pi/\mathbf{p}$,

$$(4.3) \quad Aa^{mD} - Ba^m < C_3^{1/\mathbf{p}^2} \frac{N(1-D)}{D \log a} < C_3^{1/\mathbf{p}^2} \frac{\mathbf{p}A}{2\pi C_1 D}.$$

The function $Ax^D - Bx$ attains its maximum at $x = a^t$ such that $DAx^D = Bx$, i.e.,

$$(4.4) \quad a^t = x = D^{1/(1-D)} A^{1/(1-D)} B^{-1/(1-D)},$$

and $t = \frac{1}{1-D} \log_a \frac{DA}{B}$. We will choose for m the smallest integer $m \geq t$, provided that $t > 0$.

If $DA \leq B$, then $t \leq 0$. In that case we note that $a^\varepsilon \leq e^{\pi/\mathbf{p}}$ since $N \geq 2$. Therefore, $B < e^{4/\mathbf{p}}$. We thus obtain $C_1 N D(1 - D) - \zeta(D) D(1 - D) < e^{4/\mathbf{p}}$. By Theorem 2.4, we have $1/(1 - D) < 18 \log \mathbf{p}$, hence

$$C_1 N < \frac{1}{D(1-D)} + \zeta(D) + \frac{18(e^{4/\mathbf{p}} - 1) \log \mathbf{p}}{D}.$$

The function on the right is decreasing for $1/2 \leq D < 1$. Since we can assume $\mathbf{p} > 37,000$ by Remark 1.3, we obtain $N \leq 5$.

If on the other hand $DA > B$, then $t > 0$, and we choose $m = \lceil t \rceil$. Then $a^m \geq a^t$ and $a^m < a \cdot a^t$. Also recall that $DAx^D = Bx$ and $x = a^t$. Hence

$$(4.5) \quad Aa^{mD} - Ba^m > Aa^{tD} - aBa^t = Aa^{tD}(1 - aD).$$

Combining (4.3), (4.5) and Lemma 4.4, we obtain a bound for a^{tD} . Using (4.4), this yields the following bound for A , and hence for N ,

$$(4.6) \quad N < \frac{A}{C_1(1-D)} < \frac{1}{1-D} \left(\frac{\mathbf{p}}{2\pi} \right)^{\frac{1}{D}-1} (1-D)^{1-1/D} (C_1 D)^{-1/D} e^{160 \log \mathbf{p}/\mathbf{p}}.$$

For $D = 1/2$ and $\mathbf{p} > 44,000$ (see Remark 1.3), we thus obtain

$$N < 13\mathbf{p}.$$

The function $(1 - D)^{1-1/D} (C_1 D)^{-1/D}$ is bounded away from 0 and decreasing, whereas $(1 - D)^{-1}$ increases without bound as $D \uparrow 1$. The product of these two functions decreases for $1/2 \leq D < 0.78$, and then increases. For $D < 0.96$, the value of this product is less than the value at $D = 1/2$. Thus for $1/2 \leq D < 0.96$ and $\mathbf{p} > 37,000$, we obtain

$$N < 80 \left(\frac{\mathbf{p}}{2\pi} \right)^{\frac{1}{D}-1}.$$

Close to the minimum at $D = 0.78$, for $0.69 < D < 0.86$, we even obtain

$$N < 30 \left(\frac{\mathbf{p}}{2\pi} \right)^{\frac{1}{D}-1}.$$

And for $D \geq 0.96$, we use Theorem 2.4 to obtain,

$$N < 60 \log \mathbf{p} \left(\frac{\mathbf{p}}{2\pi} \right)^{\frac{1}{D}-1}.$$

This completes the proof of Theorem 1.2. \square

Remark 4.5. Lemma 4.3 is quite weak. The integrand in (4.2) is highly oscillatory, but we only estimated the difference of the Bernoulli functions by 1, and therefore the integral by $O(\mathbf{p})$. It may be the case, however, that this integral has a fixed bound, and this would imply a uniform bound for N , independent of \mathbf{p} . This would be a highly significant result, as we now explain.

In (4.6), we estimated $C_1 N(1-D) < A$, ignoring the term $\zeta(D)(1-D)$. If N could be uniformly bounded, then taking this term into account would reduce the bound for N even more, especially for D close to 1. This could yield a zero-free region for the Riemann zeta function of the form “if $\zeta(D + i\mathbf{p}) = 0$ then $D \leq \sigma$ ” for some fixed $\sigma < 1$.

Recall from Remark 1.3 that a bound $N \leq 2$ would exclude any arithmetic progression, and $N \leq 1$ (which means a contradiction) for $D \neq 1/2$ would imply the Riemann hypothesis. This last bound is unattainable by the present methods for D close to $1/2$, because it cannot be attained at $D = 1/2$. However, if Lemma 4.3 can be improved to yield a uniform bound for N , then the present methods may be applied to the ‘doubled truncated Cantor string’, defined by

$$F(x) = \alpha T_{N,D,\mathbf{p}}(x) + T_{N,1-D,\mathbf{p}}(x)$$

where $T_{N,D,\mathbf{p}}(x) = T(x)$ is as in (1.1), and α is a positive parameter. Since the nature of this string is different for $D = 1/2$ than for $D \neq 1/2$, it might yield a result valid for all $D > 1/2$.

Clearly, for $D = 1/2$, the function F is a trivial multiple of T , whereas for $D > 1/2$, the increase of T at $x = a^m$ is reinforced in F by the extra complex dimensions at $1 - D + in\mathbf{p}$. Thus the counterpart of Lemma 4.1 would be a stronger result. On the other hand, since $\zeta(D + in\mathbf{p}) = 0$ implies $\zeta(1 - D + in\mathbf{p}) = 0$, the explicit formula for ν_F still simplifies, and the counterpart of Lemma 4.3 remains essentially the same, with an extra term containing $\zeta(1 - D)$. This term then even improves the subsequent counterpart of inequality (4.3).

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DEPARTMENT OF MATHEMATICS, UTAH VALLEY STATE COLLEGE, OREM, UT 84058-5999
E-mail address: vanframa@uvsc.edu