M1: Introduction

There are three strands that come together: the canonical pairing is about traces, the additive structure, and leads to the Fourier transform and Riemann–Roch.

Combining this with the multiplicative structure leads to the zeta function, which is the counting function of divisors on a curve, and its logarithmic derivative counts points on the curve. An inequality about the number of points is equivalent to the Riemann hypothesis for this zeta function. Bombieri proves this inequality.

The connection with the algebra of the function field is by valuations, which are described as order of vanishing at a point on the curve. Linear combinations of valuations are called divisors, which are used to describe the zeros and poles of functions. The theorem of Riemann–Roch tells you how many function there are with certain poles.

Along the way, we will see close parallels with the rational numbers and number fields. Does Bombieri’s proof give an idea how to prove the Riemann hypothesis?

The whole theory gives a fairly complete algebraic description of the geometry of a curve, but there are some gaps in the algebraic understanding, beginning with the points on the curve, which algebraically are specializations.

M2: The Canonical Pairing. The canonical pairing defines the additive character on the local fields, like

\[ e^{2\pi ixy} \]

to obtain all characters on \( \mathbb{R} \). We will see that for every local field, this works the same, once we know how to map a function to the circle group \( \mathbb{R}/\mathbb{Z} \). Note that this group is isomorphic to the circle by

\[ x \mapsto e^{2\pi ix}. \]

This will define the Fourier transform on the local fields (W1). It is important that there is one nontrivial character.
The different. The different will tell us which valuations ramify in an extension and that only finitely many valuations are ramified. An important consequence of this is that the adeles are self-dual (R1). Since every function field is an extension of \( q = \mathbb{F}_q(T) \), we need the different to identify the canonical divisor of the curve.

Additive characters. The completion of a field is a locally compact additive group. All additive characters are obtained by multiplication in the field. This provides the identification of the character group with the local field itself. The kernel of the additive character gives the local components of the canonical divisor (R1).

More geometrically, the additive characters turn out to be given by a residue computation.

The adeles are self-dual. The Fourier transform on the adeles gives us the definition of the zeta function and Riemann–Roch. This will then prove the functional equation. It is also an important ingredient in Connes’ approach.

Divisors are descriptions of the valuations of an idele.

Semi-local theory. The difference between \( E \) and \( R \) will deepen our understanding of Riemann–Roch.

Connes’ approach. Connes uses \( E \) to derive the explicit formula as a trace of the shift operator on a noncommutative space, but this goes in another direction.

Multiplicative characters. The understanding of the multiplicative structure starts with the choice of a uniformizer. The norm of the uniformizer generates the value group, as powers of \( q \), and gives the first character. (We will not consider “ramified characters.”)

Locally. The only multiplicative characters are

\[
x \mapsto |x|^s_v.
\]

These are quasi-characters: they map to \( \mathbb{C}^\ast \). The local zeta function is an integral of a function on the additive group with the additive Haar measure, against a multiplicative character. This integral converges for \( \Re s > 0 \). Surprisingly, this satisfies a functional equation.

The kernel of the multiplicative character is \( \mathfrak{o}_v^\ast \), represented by functions without a pole or zero at \( v \).
R2: Globally. The norm on the ideles plays a similar role:

$$x \mapsto |x|^s$$

gives many multiplicative maps from the ideles. The kernel is the ideles of vanishing degree.

The group \( \ker |\cdot|/K^* \) is compact, of volume \( h/(q-1) \). This is the residue it \( s = 1 \) of the zeta function.

Globally, the zeta function is defined similar to the local case, but only for \( \Re s > 1 \). This function has poles, and it satisfies a functional equation with itself. However, it requires some effort to get a formula where the domain of \( \zeta_C(s) \) overlaps with \( \zeta_C(1-s) \).

R3: The global zeta function. It will be interesting to see why the local zeta function has a functional equation of a different nature than the global zeta function.

M3: Valuations. Geometrically, these are the points and orbits of points. Every function has finitely many zeros and poles. For example, \( T \in \mathbb{F}_q \) has a zero in \( T = 0 \) and a pole in \( T = \infty \). A valuation corresponds to an orbit of Frobenius of points on the curve. Every function defined over \( \mathbb{F}_q \) will vanish to the same order at the points in an orbit, but after an extension of the finite field, more points become distinguishable.

T2: Valuations of \( \mathbb{F}_q \). These are analogous to the \( p \)-adic valuations and the archimedean valuation, a bit surprisingly because the degree is nonarchimedean. We will see in detail that these valuations correspond to points (2T2). Every valuation of a function field is an extension of one of these valuations.

T3: Constants and functions. Note that the schedule only mentions Section 1.4.1, but the introduction of Section 1.4 should be included in this.

Elements of \( \mathbb{F}_q \) are constants. Every other element is a function: an expression in \( T \) and \( X \), and putting a value for \( T \) and \( X \), say \((t,x)\), gives the value of the function at that point.

A function has as many zeros as poles. This means that \( K^* \) is contained in the kernel of the norm on the ideles.

2M2: Weil positivity. Fourier series (the Fourier transform from \( \mathbb{R}/\mathbb{Z} \) to \( \mathbb{Z} \)) provides the connection between the inequality

$$N(q) = q + O(\sqrt{q})$$

and the zeros of the zeta function. Weil positivity is equivalent to the Riemann hypothesis. Note that the coarse idele class group is
isomorphic to \( \mathbb{Z} \) and that the zeta function is a rational function in \( q^s \), hence a function on \( \mathbb{C}/2\pi i \log q \), a thickening of the circle.

**2M3: Explicit formula.** Closely related is the explicit formula for counting points on the curve. The explicit formula is a sum of Weil distributions, one for each valuation, of a test function.

**2T2: Orbits of Frobenius.** That will establish the connection between valuations and the order of vanishing at points on the curve.

Down to earth, the degree of a valuation is simply the degree of the irreducible polynomial defining the valuation. But more geometrically, this degree is the number of points in the orbit of Frobenius. For the curve, Riemann–Roch allows us to compute a nonsingular model of the curve. That such a model exists is a little involved but a fact that is easy to understand and easy to use.

**2T3: Galois covers.** These are special kinds of covers: the cover group is isomorphic with the group of field automorphisms. Surprisingly, all covers are algebraic. This is important for Bombieri’s proof.

**2W: Bombieri’s proof.** This proof has two parts. First we establish an upper bound for the point counting function. Then, using the Riemann hypothesis for \( q \) and Galois covers, we establish a lower bound. Both parts use functions in two variables (pairs of points on the curve), and the graph of Frobenius.

**2R: Nevanlinna theory.** Does Bombieri’s proof apply to power series, using Nevanlinna theory?

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**2T1: Connes’ approach**

Connes’ approach is very interesting, but a bit besides the point. But let’s see what kind of interesting considerations \( E \) leads to.

Recall for a function \( f \) on the adeles, \( E(f) \) is a function on the ideles,

\[
E(f)(a) = \sqrt{|a|} \sum_{x \in \ker | \cdot | / o^*} f(xa).
\]

If \( f(0) = \mathcal{F} f(0) = 0 \) then it satisfies the Riemann–Roch formula

\[
E(f)(a) = E(\mathcal{F} f)(1/a)
\]

for the Fourier transform on the adeles.

\( E \) provides a map from functions on \( \mathbb{A}/\ker | \cdot | \) to \( \mathbb{A}^* / \ker | \cdot | \). The first is a noncommutative space, a thickening of the integers \( \mathbb{Z} \cup \{\infty\} \) at \( \infty \). The second is an ordinary space (a group), isomorphic to \( \mathbb{Z} \).
First of all, by class field theory,

\[ \mathbb{A}^*/\ker |·| \]

is isomorphic to the subgroup of the Galois group of all abelian covers generated by Frobenius, which is the Galois group of the maximal constant field extension. \((\mathbb{A}^*/K^* \text{ corresponds to all abelian covers})\)

\(E\) has a nice property: if \(f(x) = 0\) for \(\deg x < -\omega\) then

\[ Ef(a) = 0 \]

for \(\deg a < -\omega\). Proof: for \(x \in \ker |·|/\mathfrak{p}^*\), its degree vanishes, and for \(\deg a < -\omega, \deg xa < -\omega\) (meaning that \(xa\) has many poles), and then \(f(xa) = 0\) for every such \(x\).

Further, if \(f\) is locally constant at distances in \(\pi^{-\Theta} \mathfrak{p}\) (remember \(\pi = (\pi_v)_v\) is the adele with the uniformizer at every component), then \(\mathcal{F}f\) is supported on \(\pi^{-\Theta} \mathfrak{p}\), so \(E(\mathcal{F}f)(a)\) will vanish for \(\deg a < -\theta\), where \(\theta = \deg \Theta\). If also \(f(0) = \mathcal{F}f(0) = 0\), then by Riemann–Roch, \(Ef(1/a) = E(\mathcal{F}f)(a)\) will vanish for \(\deg a < -\theta\).

Conclusion: if \(f\) is supported on \(\deg x \geq -\omega\) and \(f\) is locally constant at distances in \(\pi^{-\Theta} \mathfrak{p}\), and \(f(0) = \mathcal{F}f(0) = 0\), then \(Ef\) is supported on \(-\omega \leq \deg a \leq \theta\).

Here is one way to assure these requirements: The support of a function can be restricted by a multiplication operator. If \(\Omega\) is a function on the adeles that vanishes when \(\deg x < -\omega\), then \(M_\Omega f(x) = \Omega(x)f(x)\) vanishes when \(\deg x < -\omega\). If in addition \(\Omega(0) = 0\), then also \(M_\Omega f\) vanishes at 0. If finally \(\Omega\) only takes the values 0 and 1 then \(M_\Omega\) is a projection.

Making a function locally constant is less familiar, but is done by a convolution: if \(\theta\) is locally constant then \(C_\theta f(x) = \int_\mathbb{A} f(x - y)\theta(y)\,dy\) will be similarly locally constant. By writing this as \(\mathcal{F}M_{\mathcal{F}\theta}\mathcal{F}^{-1}\), we see similarly when this is a projection, and we can equally easily arrange that \(\mathcal{F}f(0)\) vanishes in the image.

Finally, there are clear conditions when these projections cannot commute, and surprisingly, they do commute otherwise, and the composition is then a projection. It is called the ultra violet-infra red cut-off.

The shift, cut-off in this way, does it give the explicit formula as a trace? There is some evidence that it might, but I am unable to compute it.

But let us try to do this trace computation more concretely. The calculation will be invalid, but we will see in detail how \(E\) works.
$E$ defines the zeta function:

$$q^{(g-1)s} \zeta_C(p^0, s) = q^{(g-1)s} \sum_{n=-\infty}^{\infty} E f(\phi^n) q^n\left(\frac{1}{2} - s\right),$$

where $\phi$ is an idele of degree one (norm $1/q$, corresponding to Frobenius by class field theory). This is a rational function in $q^s$ (if $f$ is locally constant and compactly supported). The coefficients of this power series (which is actually geometric) are $E f(\phi^n)$. Can we make this a finite series of coefficients (so that $\zeta_C$ would be a polynomial in $q^{-s}$)?

Closely related to this, can we get rid of the poles of $\zeta_C$?

Recall that the residue at $s = 1$ is a multiple of $F f(0)$, and at $s = 0$, a multiple of $f(0)$. By choosing $f(0) = F f(0) = 0$, we can create a zeta function without poles. Starting with $f = p^0$, choose instead $f = p^0 - p^\phi$, where $\phi$ is an idele (divisor) of degree 1. Then $f(0) = 0$.

Recall that $F f(0) = \int_A f(x) \, dx$. The function $p^0 - \frac{1}{q} p^{-\phi}$ has vanishing total mass. Combined, the function $f = \frac{1}{q} p^{-\phi} - (1 + \frac{1}{q}) p^0 + p^\phi$ has both properties. This cancels the denominator of $\zeta_C(f, s)$, and we are left with the numerator $L_C(q^{-s})$.

Let us write

$$L_C(X) = X^{2g} + l_1 X^{2g-1} + \cdots + q^0$$

for this numerator. Shifting over $\frac{1}{2}$, we have $\zeta_C(\frac{1}{2} + s)$, with a numerator with coefficients

$$1, \ l_1 q^{-1/2}, \ l_2 q^{-1}, \ldots, \ l_{2g-1} q^{\frac{g}{2} - q}, \ 1,$$

a symmetric sequence of coefficients. Thus if $\omega$ is a root then so is $1/\omega$.

In terms of sequences (say in $l^2(\mathbb{Z})$), this means that $(\omega^n)_n$ (and also $(\omega^{-n})_n$), is orthogonal to $(l_n q^{-n/2})_n$.

Let us consider the space in $l^2(\mathbb{Z})$ spanned by all shifts of $(l_n q^{-n/2})_n$. The roots $(\omega_i^n)_n$ and $(\omega_i^{-n})_n$, for $i = 1, \ldots, g$ are perpendicular to this space, and they are eigenvectors of the shift operator in this space: $S a(n) = a_{n+1}$, hence $S v = \omega v$ for $v = (\omega^n)_n$, so $S$ diagonalizes,

$$S = \begin{bmatrix} \omega_i & 0 \\ 0 & \omega_i^{-1} \end{bmatrix}$$

on the quotient $l^2(\mathbb{Z})/ \text{im}(E)$. Hence $\text{Tr}(S) = \sum_{\zeta_C(\omega) = 0} \omega$, and

$$\text{Tr}(S^n) = \sum_{\zeta_C(\omega) = 0} \omega^n.$$

This result is closely related to the explicit formula:

$$N_C(n) = q^n + 1 - \sum_{\omega} \omega^n.$$
The problem with this computation is that $(\omega^n)_n$ is not an $l^2(\mathbb{Z})$-vector, and instead, $\text{im}(E)$ is dense, and there is no orthogonal complement. To make this valid, we can introduce a cut-off, maybe consider instead the finite-dimensional space generated by $S^nL_c(X)$ for $-\omega \leq n \leq \theta$. The computations become more complicated.

2R: Nevanlinna theory

Can Bombieri’s proof be translated to a proof of the Riemann hypothesis?